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THE LOCALIZATION OF CLUSTERS OF FLOATING PARTICLES ON THE SURFACE OF TURBULENT FLOW†

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The evolution of a cluster of floating passive impurity on the surface of a turbulent flow of liquid is investigated in the one-dimensional approximation. A cluster localization effect is established which consists of the fact that the particles form an agglomeration of increased concentration with time. The evolution with time of the random width of the cluster is investigated. The physical mechanism of the localization phenomenon is discussed. © 2000 Elsevier Science Ltd. All rights reserved.

Under practical conditions, the diffusion of a floating impurity on the surface of a liquid, for example, on the ocean surface, is accompanied by chaotic motion of the impurity particles together with the liquid. Neglecting the effect of vertical motions of the liquid on the floating impurity, we will assume that the impurity particles are completely carried away by the random two-dimensional velocity field $\mathbf{v}(\mathbf{x}, t)$ – the projection of the velocity vector of the liquid onto its plane free surface [1]. Here $\mathbf{x}$ are the coordinates of points of the free surface. The surface concentration of impurity $\rho(\mathbf{x}, t)$ is then described by the convective diffusion equation

$$\frac{\partial\rho}{\partial t}+(\mathbf{v}(\mathbf{x},t)\cdot\nabla)\rho+\rho u(\mathbf{x},t)=\mu\Delta\rho$$

Here we have introduced the scalar field $u(\mathbf{x}, t) = (\nabla \cdot \mathbf{v}(\mathbf{x}, t)$ – the divergence of the two-dimensional velocity field of the liquid $\mathbf{v}(\mathbf{x}, t)$ on the free surface and μ is the molecular diffusion coefficient.

We emphasize that on the surface of even an incompressible liquid the divergence of the vector of the horizontal components of the liquid velocity, generally speaking (when the rate of change of the vertical component of the velocity directed along the normal to the surface is non-zero), is not equal to zero. Physically this means that the surface of a chaotically moving incompressible liquid behaves as a two-dimensional compressible medium. An, at first glance, unexpected effect then arises in which the clusters of floating impurity are localized, which is related to the well-known effect of the localization of waves in random multilayered media (see, for example, [2]). Some quantitative and qualitative aspects of the localization of the floating impurities were discussed in [1, 3, 4]. The related effects of occurrence of a quasi-regular coarse-scale structure in the distribution of material in the universe have been widely discussed in the astrophysical literature (see, for example, [5–8]).

In our opinion, this impurity localization effect in chaotically moving media has a universal character and is inherent in different kinds of random flows of compressible media. Below we give a quantitative description of the localization effect in the simplest case of a one-dimensional medium, which models the evolution of the concentration of a floating impurity on the surface of a liquid in a narrow channel. The results obtained many be useful when investigating and analysing localization effects in spaces of large dimensionality.

1. THE ASYMPTOTIC FORM OF THE CONCENTRATION FIELD OF FLOATING IMPURITY IN A ONE-DIMENSIONAL MEDIUM

We will discuss the behaviour of a cluster of floating particles on the surface of a turbulent flow of liquid in a channel. We will consider the idealized case when the motion of the liquid is across the channel and there is no transverse diffusion of the impurity particles. We will correspondingly assume that the field of the horizontal velocity of motion of the liquid along channel $\upsilon(x, t)$ and the concentration of floating impurity particles $\rho(x, t)$ are functions of time and only one coordinate (the longitudinal coordinate) x. The concentration of a cluster $\rho(x, t)$ then obeys the one-dimensional convective diffusion equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\upsilon(x,t)\rho)=\mu\frac{\partial^2\rho}{\partial x^2},\quad \rho(x,t=0)=\rho_0(x) \tag{1.1}$$

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To describe the random field $v(x, t)$ we will use the well-known Kraichnan approximation [9, 10] of the velocity field by a Gaussian statistically uniform field, delta-correlated in time, with a structure function

$$\langle [v(x,t) - v(x+s, t+\tau)]^2 \rangle = d(s)\delta(\tau), \quad d(s) = Bs^2 + \dots \tag{1.2}$$

We will first construct the approximate solution of Eq. (1.1) for which we write the solution in the form

$$\rho(x,t) = \overline{\tilde{\rho}(x,t)} \tag{1.3}$$

where $\tilde{\rho}(x, t)$ satisfies the first-order subsidiary equation

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x}(v(x,t)\rho) = \xi(t)\frac{\partial \tilde{\rho}}{\partial x} \tag{1.4}$$

Here $\xi(t)$ is Gaussian white noise with correlation function $\overline{\xi(t)\xi(t+\tau)} = 2\mu\delta(\tau)$, while the bar denotes averaging over the ensemble of samples $\xi(t)$.

As is well known, the solution of Eq. (1.4) can be represented in the form

$$\tilde{\rho}(x,t) = \int \rho_0(y)\delta(\tilde{X}(y,t) - x)dy \tag{1.5}$$

where $\tilde{X}(y, t)$ is the trajectory of a particle, taking molecular diffusion into account, which satisfies the stochastic equation

$$\frac{d\tilde{X}}{dt} = v(\tilde{X}, t) + \xi(t) \tag{1.6}$$

We will represent the function $\tilde{X}(y, t)$ in the form of the sum of two terms

$$\tilde{X}(y,t) = X(y,t) + z(y,t) \tag{1.7}$$

Here $X(y, t)$ describes the convective transfer of floating particles (when $\mu = 0$) while $z(y, t)$ takes into account the deviation of particle from $X(y, t)$ due to molecular diffusion and is subject to the equation

$$\frac{dz}{dt} = v(X+z, t) - v(X, t) + \xi(t), \quad z(y, t=0) = 0 \tag{1.8}$$

Suppose l_v is the characteristic scale of the spatial change in the random velocity field $v(x, t)$. We will assume that the following inequality is satisfied

$$z \ll l_v \tag{1.9}$$

which henceforth enables us, in a statistical description of the velocity field, to retain only the first non-zero term of the expansion of the structure function (1.2). If this is the case, the equation for $z(y, t)$ can be replaced by the simpler linear equation

$$\frac{dz}{dt} = u(X,t)z + \xi(t), \quad z(y, t=0) = 0, \quad u(x,t) = \frac{\partial}{\partial x}v(x,t) \tag{1.10}$$

In this linear approximation and for a velocity field $v(x, t)$ specified in advance, the process $z(y, t)$ is Gaussian with zero mean and variance $\sigma^2(y, t)$, which satisfies the equation

$$\frac{d\sigma^2}{dt} = 2u(X,t)\sigma^2 + 2\mu, \quad \sigma^2(y, t=0) = 0 \tag{1.11}$$

Substituting expression (1.7) into (1.5) and averaging the expression obtained over the Gaussian statistics of the process $z(y, t)$, we arrive at an asymptotic formula for the concentration field of the floating impurity, taking into account the convective transfer of particles and molecular diffusion

$$\rho(x,t) = \int \rho_0(y)\frac{1}{\sqrt{2\pi}\sigma(y,t)}\exp\left[-\frac{(x - X(y,t))^2}{2\sigma^2(y,t)}\right]dy \tag{1.12}$$

2. THE EVOLUTION OF THE EFFECTIVE WIDTH OF A CLUSTER OF FLOATING PARTICLES

Suppose the particles of floating impurity are situated at the point $x = 0$ at $t = 0$. Their initial concentration is then equal to $\rho_0(x) = \delta(x)$, while the concentration field (1.12) has the form

$$\rho(x,t) = R(x - X(t), t), \quad R(x,t) = \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left(-\frac{x^2}{2\sigma^2(t)}\right) \tag{2.1}$$

Here $X(t) = X(0, t)$ is the position of the centre of the cluster at the current instant of time, t, $R(x, t)$ is the concentration of the cluster in a frame of reference moving together with the cluster, and $\sigma(t) = \sigma(0, t)$ is the effective width of the cluster at the instant t.

We will investigate the behaviour of the random effective width of the cluster $\sigma(t)$ with time. It follows from stochastic equation (1.11) and also from (1.2) that $\sigma(t)$ is a Markov process, the probability distribution of which

$$f(\sigma; t) = \langle \delta(\sigma(t) - \sigma) \rangle$$

satisfies the Fokker–Planck equation [3]

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \sigma}\left(\frac{f}{\sigma}\right) = B\frac{\partial^2}{\partial \sigma^2}(\sigma^2 f), \quad f(\sigma, t = 0) = \delta(\sigma) \tag{2.2}$$

The statistical and dynamic interpretation of the consequences of this equation enables us to understand the effect of the competing action of molecular diffusion and the chaotic motion of the liquid surface on the behaviour of the cluster of floating particles. It follows from Eq. (2.2), in particular, that the root mean square of the effective width of a cluster satisfies the equation

$$\frac{d\langle \sigma^2 \rangle}{dt} = 2\mu + B\langle \sigma^2 \rangle, \quad \langle \sigma^2(t = 0) \rangle = 0$$

the solution of which

$$\langle \sigma^2(t = 0) \rangle = \int x^2 \langle R(x,t) \rangle dx = \frac{2\mu}{B}[\exp(Bt) - 1] \tag{2.3}$$

has a clear physical interpretation: at short times, so long as $Bt \ll 1$, the effect of molecular diffusion on the effective width of the cluster predominates, while the square of the effective width of the cluster increase in accordance with the classical linear relation $\langle \sigma^2(t) \rangle = 2\mu t$. Later, contraction and expansion, due to the chaotic motions of the surface of the turbulent liquid flow begin to have an effect, as a result of which $\langle \sigma^2(t) \rangle$ begins to increase more rapidly, and at times $Bt \ll 1$ the linear growth changes to an exponential growth.

Subsequently, when the value of $\langle \sigma^2(t) \rangle$ becomes comparable with the square of the characteristic scale l_v of the velocity $\upsilon(x, t)$ of the turbulent motion of the liquid, the motion of the particles can be assumed to be practically independent of one another, formula (2.3) becomes inapplicable, and the effective width of the cluster again increases linearly [11] $\langle \sigma^2(t) \rangle = 2(D + \mu)t$, where D is the turbulent diffusion coefficient.

Note further that the dimensionless parameter, the smallness of which ensures that asymptotic solution (2.1) and all subsequent calculations hold, is the ratio

$$\varepsilon = 2\mu/(Bl_v^2)$$

i.e. the ratio of the coefficient on the right-hand side of (2.3) to the square of the characteristic scale of turbulence.

Note that the conclusions drawn regarding the behaviour of the cluster, based on an analysis of the root mean square of the effective width of the cluster (2.3) and its other moments, may not reflect the true behaviour of the cluster with time (see, for example, [1]). A more accurate analysis must be based on a discussion of the probability properties of the effective width of the cluster.

We can convince ourselves of this by investigating some probability consequences of Eq. (2.2). In Fig. 1 we show the results of a numerical solution of this equation with the initial condition $fl(\sigma/l;$

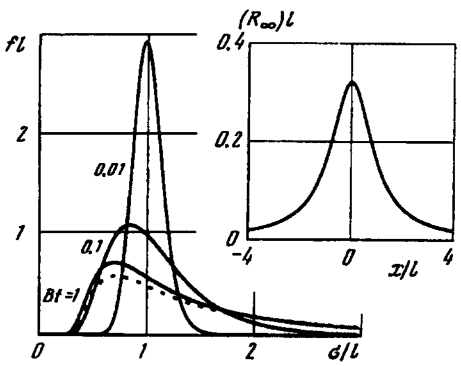

Fig. 1.

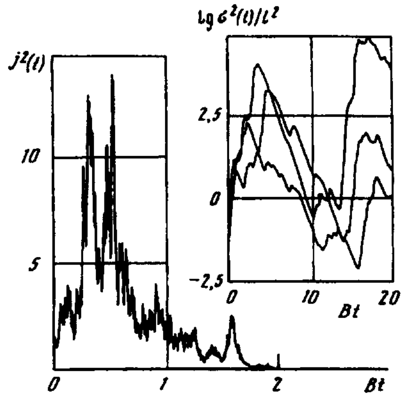

Fig. 2.

$t = 0) = \delta(\sigma/l - 1)$, where $l = \sqrt{(\mu/B)}$ is a parameter which has the dimension of length. It can be seen that the solution approaches a steady distribution (the dashed curve in Fig. 1), describing the statistics of the effective width of the cluster for long times. This steady distribution can be obtained analytically

$$f_\infty(\sigma) = \lim_{t\to\infty} f(\sigma; t) = \sqrt{\frac{2}{\pi}} \frac{l}{\sigma^2} \exp\left(-\frac{l^2}{2\sigma^2}\right) \tag{2.4}$$

The presence of steady distribution (2.4) indicates that the turbulent motion of the medium mainly weakness the molecular diffusion and leads to localization of the cluster. In fact, by averaging the cluster concentration (2.1) using probability distribution (2.4), we obtain that the average profile of the concentration is quite well localized on the x axis (see also the graph on the right in Fig. 1)

$$\langle R_\infty(x)\rangle = \lim_{t\to\infty}\langle R(x, t)\rangle = \frac{1}{\pi}\frac{l}{x^2 + l^2} \tag{2.5}$$

We emphasize, however, that this expression does not contradict the fact that the root mean square effective width of the cluster (2.3) tends to infinity as $t \to \infty$ since the integral defining it (the first equation of (2.3)) diverges.

Localization of the cluster also leads to the fact that the maximum concentration value

$$\rho = R(x = 0, t) = 1/(\sqrt{2\pi}\sigma(t))$$

does not tend monotonically to zero with time, as in the case of Brownian floating particles on the surface of a liquid at rest, and has the steady distribution

$$W(\rho) = \frac{2}{\sqrt{\pi} r} \exp\left(-\frac{\rho^2}{r^2}\right), \quad r = \frac{1}{l\sqrt{\pi}}$$

3. THE PHYSICAL MECHANISM OF THE CLUSTER LOCALIZATION EFFECT

We will examine in more detail the mechanism by which clusters of floating particles are localized on the surface of a chaotically moving medium. To do this we note that quantity $z(x) = z(0, t)$ has a clear physical meaning. This random departure of a certain particle of the cluster from its centre is closely related to the dispersion function (the Jacobian of the transition from Lagrange y-coordinates to Euler x-coordinates)

$$J(t) = \partial X/dy$$

which satisfies the following equation in the one-dimensional case considered

$$dJ/dt = u(X, t)J \tag{3.1}$$

In fact, the solution of stochastic equation (3.1)

$$z(t)=\int_0^t \xi(t')J(t,t')dt' \tag{3.2}$$

where $J(t, t')$ is the solution of Eq. (3.1) with initial condition $J(t = t') = 1$, has a clear geometrical interpretation: expression (3.2) is the sum of the generating Brownian motion of "microscopic impacts" $\xi(t')$, multiplied by the degree of contraction (when $J < 1$) or expansion ($J > 1$) of the cluster of particles. In the case of a liquid at rest $J \equiv 1$, while integral (3.2) describes classical Brownian motion. In the general case, the motion of the liquid, which leads to random contractions and expansions of the cluster, may qualitatively change the cluster particle diffusion process. A similar assertion holds with regard to $\sigma^2(t)$ – the variance of the departure of cluster particles from its centre. By relations (1.11) and (3.1) it is equal to

$$\sigma^2(t)=2\mu\int_0^t J^2(t,t')dt' \tag{3.3}$$

and is also determined by the behaviour of the function $J(t, t')$.

We will discuss its properties in more detail. In the delta-correlated approximation of the velocity field $\upsilon(x, t)$ considered, the function $J(t, t')$ is a Markov process, the probability distribution of which

$$f(j;t,t')=\langle\delta(J(t,t')-j)\rangle$$

satisfies the Fokker–Planck equation [1, 3, 4]

$$\frac{\partial f}{\partial t}=B\frac{\partial^2}{\partial j^2}(j^2 f),\quad f(j;t=t',t')=\delta(j-1) \tag{3.4}$$

We will investigate the behaviour of samples of $J(t, t')$, for which we will use the method proposed previously [3], namely we will identify $J(t, t')$ with a statisticality equivalent process, which satisfies the model stochastic equation

$$\frac{dJ}{dt}+BJ=\eta(t)J,\qquad J(t=t',t')=1 \tag{3.5}$$

where $\eta(t)$ is white noise with correlation function $\langle\eta(t)\eta(t + \tau)\rangle = 2B\delta(\tau)$. The basis of replacing $J(t, t')$ by a statistically equivalent process is the fact that the probability distribution of the solution of stochastic equation (3.5) satisfies Eq. (3.4). In this case it is natural to expect that the properties of samples of the process $J(t, t')$ and the statistically equivalent process will be the same.

We draw attention to the last term on the left-hand side of Eq. (3.5), since it is precisely this term that is responsible for the localization of clusters of floating impurity. This term takes into account the fact that, in the case of a Gaussian field $\upsilon(x, t)$ with structure function (1.2), the average of the divergence of the velocity field at the cluster centre, when $x = X(t)$, is negative [3, 4]; $\langle u(X(t), t)\rangle = -B$. This equality indicates that random horizontal motions of the liquid surface often contracts the cluster rather than expand it, which also leads in a finite time to localization.

The solution of Eq. (3.5) has the form

$$J(t,t')=\exp(-\theta+\omega(t,t')),\quad \omega(t,t')=\int_{t'}^{t}\eta(\tau)d\tau,\quad \theta=B(t-t') \tag{3.6}$$

where $\omega(t, t')$ is a Wiener process with a variance of 2θ.

Samples of the process $J(t, t')$ (3.6) possess superficial contradictory statistical properties. Its moments

$$\langle J^n(t,t')\rangle=\exp(n(n-1)\theta) \tag{3.7}$$

when $n > 1$ grow exponentially with time. On the other hand, the log-normal probability distribution of the process $J(t, t')$

$$f(j;t,t')=\frac{1}{2\sqrt{\pi\theta}}\exp\left(-\frac{\ln^2(je^{\theta})}{4\theta}\right)$$

has a median $j_{max} = \exp(-\theta)$, which approaches zero exponentially. This denotes that the random quantity $J(t, t')$ takes values close to $j = 0$ with time with greater and greater probability (this combination of contradictory statistical properties is characteristic for intermittent processes [12, 13]).

The exponential increase in the moments (3.7) is due to the presence of high peaks in some samples of $J(t, t')$. On the other hand, a large part of the samples of $J(t, t')$ approaches zero with time. This conclusion is confirmed by the presence of majorant curves which fall exponentially to zero [1–3, 12]

$$M(t) = A \exp(-rBt), \quad A > 1, \quad 0 < r < 1 \tag{3.8}$$

below which there are samples of the process $J(t, t')$. The latter property of the samples of $J(t, t')$ ensures the existence of the steady distribution (2.4) of the random effective width of the cluster and the occurrence of localization of the cluster due to competition between molecular diffusion and convective chaotic motion of the particles of floating impurity.

Figure 2 shows a typical graph of a sample of the process $J^2(t) = J^2(t, t' = 0)$, which illustrates the behaviour of the integrand in expression (3.3), which defines the evolution of the random effective width of a cluster. It can be seen that $J^2(t)$ falls to zero with time, leading to asymptotic stationarity of the effective width of a cluster, which the graphs of the time dependence of three samples of the effective width of a cluster, shown on the right of Fig. 2, demonstrate (we have chosen a logarithmic scale along the vertical coordinate because of the large spread in the values of $\sigma(t)$ in different samples).

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REFERENCES

1. KLYATSKIN, V. I. and SAICHEV, A. I., The statistical theory of the diffusion of floating impurity in a random velocity field. *Zh. Eksp. Teor. Fiz.*, 1997, **111**, 4, 1297–1313.
2. KLYATSKIN, V. I. and SAICHEV, A. I., Statistical and dynamic localization of plane waves in chaotically multilayered media. *Uspekhi Fiz. Nauk*, 1992, **162**, 3, 161–194.
3. SAICHEV, A. I. and WOYCZYNSKI, W. A., Probability distribution of passive tracers in randomly moving media. *Stochastic Models in Geosystems* (The IMA Volumes in mathematics and its applications). Springer, New York, 1996. **85**, 359–399.
4. SAICHEV, A. I. and ZHUKOVA, I. S., The arising and evolution of the passive tracer clusters in compressible random media: *Lecture Notes in Physics*. Springer, Berlin, 1998, **511**, 2, 353–371.
5. SHANDARIN, S. F. and ZELDOVICH, Ya. B., The large-sclae structure of the Universe. Turbulence, intermittency, structures in a self-gravitating medium. *Rev. Modern Phys.*, 1989, **61**, 2, 185–220.
6. GURBATOV, S. N., MALAKHOV, A. N. and SAICHEV, A. I., *Non-linear Random Waves in Dispersion-Free Media*. Nauka, Moscow, 1990.
7. WEINBERG, D. H. and GUNN, J. E., Large scale structure and the adhesion approximation. *Monthly Not. Roy., Astron. Soc.* 1990, **247**, 2, 260–286.
8. SHANDARIN, S. F. and SATHYAPRAKASH, B. S., Modeling gravitational clusters without computing gravitation force. *Astrophys. J.* 1996, **467**, 1, Pt 2, L25–L28.
9. KRAICHNAN, R. H., Small-scale structure of a scalar field convected by turbulence. *Phys. Fluids*., 1968, **11**, 5, 945–953.
10. KRAICHNAN, R. H., Anomalous scaling of a randomly advected passive scalar. *Phys. Rev. Lett.* 1994, **72**, 7, 1016–1019.
11. TEODOROVICH, E. V., Turbulent diffusion of a passive impurity. *Prikl. Mat. Mekh.*, 1991, **55**, 2, 1 275–282.
12. ZEL'DOVICH, Ya. B., MOLCHANOV, S. A., RUZMAIKIN, A. A. and SOKOLOV, D. D., Intermittency in a random medium. *Uspekhi Fiz. Nauk*, 1987, **152**, 1, 3–32.
13. FRISCH, U., *Turbulence. The Legacy of A.N. Kolmogorov*. Cambridge University Press, Cambridge, 1996.

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